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MOMENT INVARIANTS FOR PARTICLE BEAMS*

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ABSTRACT

The rms emittance is a certain function of second moments in 2-D phase space. It is preserved for linear uncoupled (1-D) motion. In this paper, we present new functions of moments that are invariants for coupled motion. These invariants were computed symbolically using a computer algebra system. Possible applications for these invariants are discussed. Also, approximate moment invariants for nonlinear motion are presented.

INTRODUCTION

Moments and Moment Equations

Let $g(\vec{x},\vec{p},t)$ be a function defined on six-dimensional phase space. We define the moment of g to be the integral of g over all of phase space, weighted by the single-particle phase-space distribution function. That is,

$$< g(\vec{x},\vec{p},t) > = \int_{-\infty}^{\infty} d^3p \int_{-\infty}^{\infty} d^3x \, f(\vec{x},\vec{p},t)\, g(\vec{x},\vec{p},t), \tag{1}$$

where f is the distribution function. For a discrete distribution consisting of N particles, with phase-space coordinates $\vec{x}_i, \vec{p}_i$, we have

$$< g(\vec{x},\vec{p},t) > = \frac{1}{N}\sum_{i=1}^{N} g(\vec{x}_i,\vec{p}_i,t). \tag{2}$$

This form is useful in computing moments in particle simulation codes. Note that these moments are functions of time only. A particle beam can be described by a set of moments of some basic functions such as the monomials in the variables x, p_x, y, p_y, z, p_z. The moment description has the advantage of being closely related to laboratory quantities. For example, the centroid of a beam in the x-direction is given by $< x >$ and the rms width in the x-direction is given by $\sqrt{< x^2 >}$.

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For particle beams in most noncircular machines, the distribution function accurately satisfies the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \vec{x}} \cdot \frac{\vec{p}}{m} + \frac{\partial f}{\partial \vec{p}} \cdot \vec{F} = 0, \tag{3}$$

where $\vec{F}$ is the sum of the external force and the space-charge force, whose potential ϕ satisfies the Poisson equation

$$\nabla^2 \phi = -\frac{1}{\epsilon_o} \int_{-\infty}^{\infty} d^3p\, f(\vec{x}, \vec{p}, t). \tag{4}$$

If we are describing the beam with moments, we need a way to relate the integral in the above formula to the moments. (This is the major problem in any simulation scheme using moments as the dynamical variables. We will not be concerned with this here, however.) The rule for differentiating moments with respect to time is given by[1]

$$\frac{d}{dt} < g > = < \frac{d}{dt} g >, \tag{5}$$

which says that the derivative of a moment is the moment of a derivative. Therefore, it is easy to get the equations of motion for the moments. For example, the equation for the $< xp_x >$ moment is

$$\begin{aligned} \frac{d}{dt} < xp_x > &= < (\frac{d}{dt} x) p_x > + < x (\frac{d}{dt} p) > \\ &= \frac{1}{m} < p_x^2 > + < x F_x > \\ &= \frac{1}{m} < p_x^2 > + < x(a_0 + a_1 x + a_2 x^2 + \cdots) > \\ &= \frac{1}{m} < p_x^2 > + a_0 < x > + a_1 < x^2 > + a_2 < x^3 > + \cdots . \end{aligned} \tag{6}$$

Because we expanded the force into a power series, the right hand side of the equation contains only moments. So the moment equations are a system of linear differential equations except that, in the case with space charge, the force coefficients will depend on the spatial moments.

Emittance and RMS Emittance

Emittance can be defined in various ways, but it is always some kind of area in a 2-D projection of phase space. If the x-degree of freedom is uncoupled from the y and z directions, then the total emittance in the x-direction is a conserved quantity because it is the same as the phase volume. It is an invariant that does not depend on the force.

The rms emittance in the x-direction is defined to be the following function of second moments:

$$\epsilon_x = \sqrt{< x^2 >< p_x^2 > - < xp_x >^2}. \tag{7}$$

This definition assumes that the beam centroid is at the origin. Subtracting $< x >$ and $< p_x >$ from x and p_x, respectively, in the above definition will give the general formula. The rms emittance is conserved for linear, uncoupled motion. To see this, simply differentiate ϵ_x with respect to time, assuming the force is proportional to x, that is, assuming $dp_x/dt = a_1 x$.

For uncoupled motion, the rms emittance is somewhat like the total emittance in that it is also an invariant that does not depend on the force (value of constant a_1). But it is invariant only for linear forces. For coupled motion, there are no know invariants that are functions of emittances, even for linear forces. The problem we solve in this paper is to find other moment invariants, analogous to the rms emittance, that are valid for more general situations than linear, uncoupled motion.

FINDING NEW INVARIANTS

Setting Up the Problem

Consider the following homogeneous quadratic function of the second moments in one dimension (the x-direction).

$$\begin{aligned} I &= c_{11} < x^2 >< x^2 > + c_{12} < x^2 >< xp_x > + c_{13} < x^2 >< p_x^2 > \\ &+ c_{21} < xp_x >< x^2 > + c_{22} < xp_x >< xp_x > \\ &+ c_{31} < p_x^2 >< x^2 > \end{aligned} \tag{8}$$

For the following choice of the coefficients,

$$\begin{aligned} c_{13} &= 1, \\ c_{22} &= -1, \\ \text{other } c_{ij} &= 0, \end{aligned} \tag{9}$$

the function I is the square of the rms emittance in the x-direction and is therefore an invariant for linear, uncoupled motion.

Let us now find the moment invariant for linear motion in 2-D. To do this, let us start with an expression like Eq. (8) but include all second moments involving both the x and y directions. Then we will attempt to find a set of values for the coefficients that makes the expression an invariant.

First define the vector

$$\begin{aligned} X = (&< xx >, < xp_x >, < xy >, < xp_y >, \\ &< p_x x >, < p_x p_x >, < p_x y >, < p_x p_y >, \\ &< yx >, < yp_x >, < yy >, < yp_y >, \\ &< p_y x >, < p_y p_x >, < p_y y >, < p_y p_y >). \end{aligned} \tag{10}$$

This is a list of all the possible moments that our invariant can contain. Then the proposed invariant can be written as

$$I = X^T C X. \tag{11}$$

Remarkably enough, in writing Eq. (11), we have already sufficiently restricted the form of the invariant so that by applying the condition

$$\frac{d}{dt} I = 0, \tag{12}$$

we can solve for the coefficient matrix C and thereby determine the desired invariant. In taking the time derivative, we must assume that the forces are given by

$$\begin{aligned} F_x &= x a_{xx} + y a_{xy}, \\ F_y &= x a_{xy} + y a_{yy}. \end{aligned} \tag{13}$$

Notice the symmetry of the force coefficients, which is equivalent to requiring that the force be derivable from a potential. (Without this restriction, the invariants do not exist.)

Because of the size of this problem (we have to solve for 55 coefficients), we used the SMP (Symbolic Manipulation Program) computer algebra system to do the algebra symbolically.

Using the SMP Computer Algebra System

The computer algebra system SMP is a product of Inference Corporation (Los Angeles, CA). It is an interactive system that takes commands at the terminal and returns results on the screen. It can also read files containing commands. In this section, we show some general features of this system. In the next section, we will show how our actual problem was solved.

In SMP, most objects such as functions, matrices, and parts of expressions are all represented by things called projections. For example, if `p` is a polynomial, then the projection `p[3]` is the third term. To see how SMP works, consider an object `f`. Now define some properties of `f`. If we think of `f` as an array, then the command

```
f[1,4]:6
```

defines the (1,4)-th element to have value 6. Now add another property to `f` by issuing the following command:

```
f[x]:7
```

This causes `f` to map the quantity x into the value 7. If we now enter the command

```
f[$x]:$x^2
```

we define the function $f(x) = x^2$. (The $x is a dummy argument and is said to be generic.) Now when f is evaluated, an argument of 1,4 returns 6, an argument of x returns 7, and any other argument returns the square of the argument. So, for example, the value of f[x] is 7, but the value of f[y] is y^2.

More complicated definitions are possible. The following

```
f[$x _= ($x>7 | $x<-9)] : 75
```

says that if the argument is greater than 7 or less than -9, then the value of f is 75. The symbol _= should be thought of as "such that." Existing built-in functions can be redefined. For example, the following

```
D[h[$x],{$x,1,$y}]:g[$y]
```

adds to the differentiation function D the rule that the derivative of the function h is g. The list {$x,1,$y} means that we are taking the first derivative with respect to $x and evaluating at $y. After we make this definition, D[x^2,x] returns 2x so the old (built-in) definition is still there. But D[h[z],z] returns g[z], using the new rule.

THE NEW LINEAR INVARIANTS

To solve the problem of the invariant for the linear coupled motion, we had to define many SMP functions. We show some of these here. In the SMP code, the moment $< g >$ is denoted by mom[g]. The linearity property of moments is defined by the following SMP code.

```
mom[$x + $$y] :: mom[$x] + mom[$$y]
mom[$$y $x _= test[$x]] :: $x mom[$$y]
```

The $$y symbol is said to be doubly generic and stands for a complete expression, not just a single term. The function test is defined to return a true value if the argument is a constant. The following definitions cause <g> in the input commands to be translated into mom[g] and print out mom[g] as <g>.

```
"<" := "mom["
">" := "]"
_mom[Pr,$x] : Fmt[,"<",$x,">"]
```

This makes it possible to communicate with SMP using familiar notation.

We define moments to be the list of variables. Right now, we are doing the 2-D case, so we define this list as follows.

```
moments : {x,px,y,py}
```

Now define **der** to be the function take takes a time derivative. In addition to giving it the usual rules for differentiating possesed by the built-in D function, we add the rules for taking derivatives of coordinates.

```
der[x] :: px
der[y] :: py
der[px] :: fx
der[py] :: fy
```

Now we give the rule that the derivative of the moment is the moment of the derivative.

```
der[mom[$x]] :: mom[der[$x]]
```

We define a list of second moments, called **vector** by the following piece of SMP code.

```
vector2:Ldist[mom[Union[Flat[moments**moments]]]]
vector:Union[vector2]
```

This takes the outer product of **moments**, flattens this matrix into a list, and converts this list into a list of moments of these quantities.

Now letting **cmat** be some matrix, we define **eq**, the proposed invariant.

```
eq::Ex[vector.cmat.vector]
```

This is the same as Eq. (11), which says that our proposed invariant is a homogeneous quadratic combination of second moments. After defining the forces,

```
fx : x a[x,x] + y a[x,y]
fy : x a[x,y] + y a[y,y]
```

we differentiate our proposed invariant with the **der** function. Setting the coefficients of the moments and forces individually to zero leads to a linear system of equations for the coefficients. Using a built-in function in SMP to solve this system yields the coefficients in the invariant. Substituting these coefficients into the expression for **eq** gives us the desired invariant. The result, which we call I_2, is the following:

$$I_2 = \epsilon_x^2 + 2\epsilon_{xy}^2 + \epsilon_y^2, \tag{14}$$

where

$$\begin{aligned} \epsilon_x^2 &= <x^2><p_x^2> - <xp_x>^2, \\ \epsilon_y^2 &= <y^2><p_y^2> - <yp_y>^2, \\ \text{and}\quad \epsilon_{xy}^2 &= <xy><p_xp_y> - <xp_y><yp_x>. \end{aligned} \tag{15}$$

Equation (14) is our main result. It says that for two dimensions, the invariant is the sum of the square of the x-rms emittance, the square of the y-rms emittance, and a third term, which could be called a "cross-emittance." There is a similar result for three dimensions.

We also found that there are other invariants for linear motion. If we search for invariants that are homogeneous quadratic functions of *fourth* moments, we obtain (for 1-D)

$$I_4 = < x^4 >< p_x^4 > -4 < x^3 p_x >< x p_x^3 > +3 < x^2 p_x^2 >^2. \tag{16}$$

For sixth moments, we obtain

$$\begin{aligned} I_6 \;&=\; < x^6 >< p_x^6 > -6 < x^5 p_x >< x p_x^5 > \\ &\quad + 15 < x^4 p_x^2 >< x^2 p_x^4 > -10 < x^3 p_x^3 >^2. \end{aligned} \tag{17}$$

There are apparently an infinite number of such invariants. There are 1-D, 2-D, and 3-D versions of all of them.

There is a very simple way to generate all these invariants. They are all of the form

$$I_n \;=\; \frac{1}{2} < (X - P)^n >_{sym}, \tag{18}$$

where

$$\begin{aligned} X \;&=\; x + y + z, \\ P \;&=\; p_x + p_y + p_z, \end{aligned} \tag{19}$$

and where the "symmetrized" moment is defined by its linearity properties

$$\begin{aligned} < g + h >_{sym} \;&=\; < g >_{sym} + < h >_{sym} \\ < \alpha g >_{sym} \;&=\; \alpha < g >_{sym} \end{aligned} \tag{20}$$

and by its action on monomials

$$< x^i p_x^j \; y^k p_y^l \; z^m p_z^n >_{sym} = < x^i p_x^j \; y^k p_y^l \; z^m p_z^n >< x^j p_x^i \; y^l p_y^k \; z^n p_z^m > . \tag{21}$$

This says that the symmetrized moment is just the usual moment, multiplied by a moment in which we interchange the powers of the coordinates and the conjugate momenta. The following, which computes the lowest-order invariant in 1-D, shows how this works.

$$\begin{aligned} 2I_2 \;&=\; < (x - p_x)^2 >_s \\ &=\; < x^2 - 2x p_x + p_x^2 >_s \\ &=\; < x^2 >_s - 2 < x p_x >_s + < p_x^2 >_s \\ &=\; < x^2 >< p_x^2 > -2 < x p_x >< p_x x > + < p_x^2 >< x^2 > \\ I_2 \;&=\; < x^2 >< p_x^2 > - < x p_x >^2 \end{aligned} \tag{22}$$

NONLINEAR INVARIANTS

Consider some distribution in (x, p_x) phase space. Now consider a linear transformation to a new coordinate system $(\bar{x}, \bar{p}_x)$ in which the moment $< \bar{x}\bar{p}_x >$ vanishes. The rms emittance for this distribution is given by

$$\epsilon_x = \sqrt{< x^2 >< p_x^2 > - < xp_x >^2} = \sqrt{< \bar{x}^2 >< \bar{p}_x^2 >}. \tag{23}$$

In the new coordinates, the rms emittance is the product of the rms widths in the $\bar{x}$- and $\bar{p}_x$-directions. It is like an area in phase space. The cross term $- < xp_x >^2$ in the definition of the rms emittance serves to take out the effect of the "tilt" in the phase-space distribution.

If nonlinear forces are present, then the "twist" of the distribution can change. For example, if the initial distribution is a line in (x, p_x) space, the action of nonlinear forces will introduce an S-shape into the line. This causes the rms emittance to grow. If we seek a moment invariant that is preserved in the presence of nonlinear forces, then we need terms like $< xp_x^3 >$ in the invariant that will take out the effect of the twist in the distribution. The higher invariants I_n do contain such terms; however, as we shall see, these are not nonlinear invariants.

But let us look for nonlinear invariants that are combinations of the I_n. Consider the following candidate for a nonlinear invariant.

$$J = I_4 + cI_2^2 \tag{24}$$

Because it is a function of linear invariants, J is constant for linear motion regardless of the value of c. Let us try to choose a value of c that will make J a nonlinear invariant. To do this, consider the distribution in (x, p_x) phase space that is constant in the region

$$\begin{array}{rcccl} -x_{max} & \leq & x & \leq & x_{max} \\ -\Delta p_x & \leq & p_x - (\alpha x + \beta x^3) & \leq & \Delta p_x \end{array} \tag{25}$$

and zero elsewhere. This distribution is shown in Fig. 1. This distribution has four parameters: x_{max}, Δp_x, α, and β. In order to get simplify the final result, let us eliminate the parameters Δp_x and β in favor of the rms emittance ϵ_x and another parameter δ, which measures the amount of twist in the distribution.

$$\Delta p_x = \frac{3\,\epsilon_x}{x_{max}}, \qquad \beta = \sqrt{\frac{525\,\delta\epsilon_x^4}{2\,x_{max}^4}} \tag{26}$$

Now compute the linear invariants I_2 and I_4 for the distribution parametrized by x_{max}, ϵ_x, α, and δ. This can be done using SMP to symbolically compute the required integrals over the distribution given by Eq. (25).

$$I_2 = \epsilon_x^2 + \delta^2\epsilon_x^8 \tag{27}$$

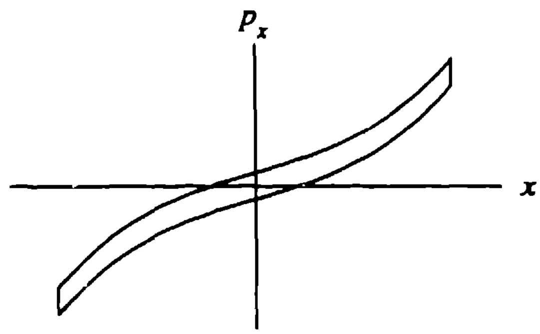

Fig. 1. This phase-space distribution was used to obtain a nonlinear invariant by requiring that the value of the invariant did not depend on the amount of "twist" in the distribution.

$$I_4 = \frac{156}{25}\epsilon_x^4 + 20\ \delta^2\epsilon_x^{10} + \mathcal{O}(\delta^4) \tag{28}$$

Notice that these linear invariants depend only on the rms emittance ϵ_x and on the parameter δ, which measures the amount of twist in the distribution. These invariants do not depend on α, the amount of tilt in the distribution, nor on x_{max}, which measures the eccentricity of the distribution. Substituting the above I_2 and I_4 in the definition for J, we see that if $c = -10$, then J is independent of the twist parameter δ. So the desired invariant is

$$J = I_4 - 10 I_2^2. \tag{29}$$

The utility of this invariant was verified numerically for a situation in which a uniformly filled ellipse in phase space was transformed by a system containing linear and cubic forces. Figure 2 shows the initial and final phase-space distributions for our example. The strength of the nonlinearity is sufficient to put a noticeable twist in the distribution, which initially did not have any. Figure 3 shows some of the linear invariants and the new J invariant as functions of time, showing their evolution between the initial and final states shown in Fig. 2. In this graph, all invariants are normalized by their initial values. We see that the linear invariants are not constant and that the higher invariants are even more sensitive to the nonlinearity than is I_2. The new invariant J is more constant

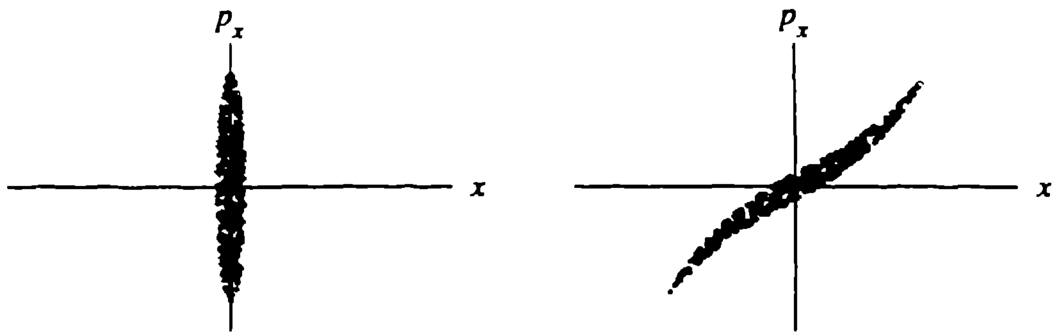

Fig. 2. The nonlinear invariant was checked by simulating the evolution of the initial distribution shown on the left in the presence of linear and cubic forces. The final distribution is shown on the right. The strength of the nonlinearity, for the given beam size and the elapsed time chosen, is large enough to introduce a noticeable twist in the distribution.

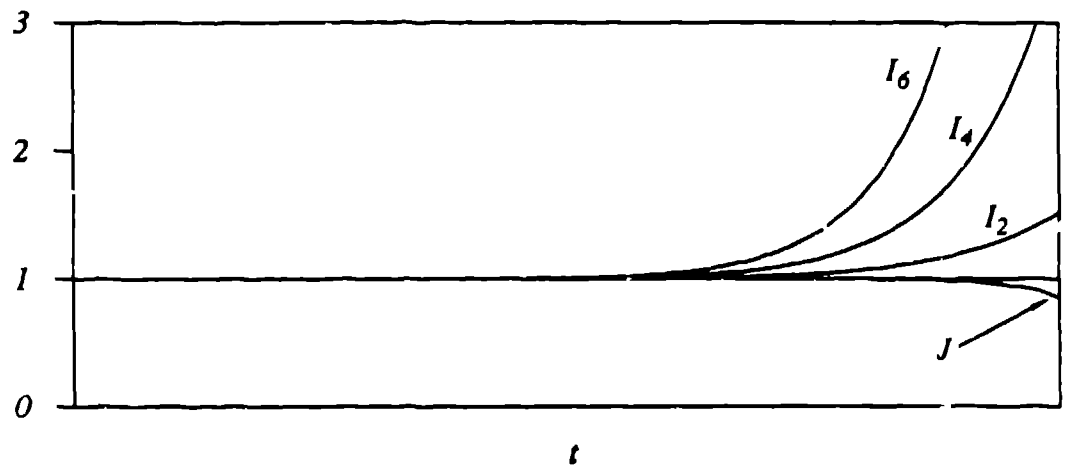

Fig. 3. The linear invariants I_n and the approximate nonlinear invariant J are shown as functions of time for the situation described in Fig. 2. All invariants are normalized by their initial values. Even though the quantity J is a linear combination of I_4 and the square of I_2, it is approximately constant.

than the rest, but it is not an exact invariant.

DISCUSSION

The moment description is useful because it is so closely related to laboratory quantities. The well-known rms emittances ϵ_x, ϵ_y, and ϵ_z are functions of second moments.

$$\begin{aligned} \epsilon_x^2 &= \langle x^2 \rangle\langle p_x^2 \rangle - \langle xp_x \rangle^2 \\ \epsilon_y^2 &= \langle y^2 \rangle\langle p_y^2 \rangle - \langle yp_y \rangle^2 \\ \epsilon_z^2 &= \langle z^2 \rangle\langle p_z^2 \rangle - \langle zp_z \rangle^2 \end{aligned} \tag{30}$$

These quantities are invariant for linear, uncoupled (1-D) motion. If an accelerator or beamline has any coupling forces, even if they are linear, the rms emittance can change. Thus, the growth in an rms emittance is not necessarily an indication of nonlinearities in the machine.

We have shown in this paper a new moment invariant, I_2, that is conserved for linear forces even in 2- or 3-D. In 1-D it is the same as square of the rms emittance. In 2- or 3-D it is the sum of the squares of the rms emittances together with new cross terms. In three dimensions, we have

$$\begin{aligned} I_2 = \quad & \langle x^2 \rangle\langle p_x^2 \rangle - \langle xp_x \rangle^2 \\ + \; & \langle y^2 \rangle\langle p_y^2 \rangle - \langle yp_y \rangle^2 \\ + \; & \langle z^2 \rangle\langle p_z^2 \rangle - \langle zp_z \rangle^2 \\ + \; & 2\langle xy \rangle\langle p_xp_y \rangle - 2\langle xp_y \rangle\langle yp_x \rangle \\ + \; & 2\langle xz \rangle\langle p_xp_z \rangle - 2\langle xp_z \rangle\langle zp_x \rangle \\ + \; & 2\langle yz \rangle\langle p_yp_z \rangle - 2\langle yp_z \rangle\langle zp_y \rangle . \end{aligned} \tag{31}$$

This invariant, or any of our other new invariants, is easily obtained using the formula given by Eq. (18). In this case, we just expanded

$$\langle (x + y + z - p_x - p_y - p_z)^2 \rangle_{sym}.$$

The utility of the new invariant is that it is conserved for any kind of linear motion, even with coupling present. Therefore, if a simulation or measurement showed that the quantity I_2 has increased, then we can conclude that the accelerator or beamline contains nonlinearities.

We have also found other moment invariants for linear motion that are homogeneous quadratic combinations of higher moments; these are the quantities I_2, I_4, I_6, etc. All these exist in 1-D, 2-D, and 3-D versions.

Consider, for a moment, uncoupled linear motion. We know that the beam current and the rms emittance are the only significant beam parameters in this situation*. Suppose we have some beam and some beamline. All we have to know is whether the beamline can transport the emittance of our beam. Neither the details of the machine acceptance nor of our beam are important. We know that we can always make a linear matching section to match our beam into the machine. This is true because any beam of a given emittance can be transformed by a linear transformation into any other beam of the same emittance.

The above is only true if our knowledge is limited to second moments. That is, we consider two beams identical if they have the same collection of moments, up to second order. If we describe our beam to higher order, then we need new invariants. This is how the higher invariants I_n presented in this paper can be used. For example, if we describe a beam by moments up to fourth order, then the invariant beam parameters are I_2 and I_4. (I_2 is the square of the rms emittance, of course.)

In the more general situation of linear coupled motion, the same ideas apply, but we need additional invariants. In a second-order description in 2-D, we need two invariants, but we have only one, the 2-D version of I_2. We need further work to determine these invariants. Before we discuss extending these results to nonlinear motion, let us consider other possible uses for the linear invariants.

In numerical simulations, it is useful to use slow variables. For example, for nonlinear oscillatory single-particle motion, it is better to use amplitude-phase variables rather than x, p_x because we are solving directly for the effect of the nonlinearity. If the nonlinearity is zero, then the amplitude and phase are constant. In the same way, we could use our moment invariants as the dynamical variables in a simulation code. To do this, we have to determine all the independent invariants up to a given order. In linear single-particle motion there is a simple relation between the amplitude-phase variables and the action-angle variables. If the new moment invariants are like the action variables, perhaps we can find a Hamiltonian formulation of the moment equations. Therefore, we believe the linear moment invariants may have significance both in numerical simulation and in theoretical analysis of beam motion.

*The beam current depends on the charge in the beam, which is the zeroth phase-space moment and could be considered to be the invariant I_0 in our scheme. We prefer to not consider here moments below second order and to consider space charge as a source of forces in addition to those of the beamline elements.

Because the linear moment invariants respond only to nonlinearities, they are useful in studying nonlinear motion as well as linear motion. However, determining nonlinear invariants would be even more useful. For example, we would like an invariant that is constant if no fifth- or higher-order forces are present. Such invariants would help us to analyze aberrations. We found the approximate nonlinear invariant J by assuming a certain model for a beam. The difficulty with working with nonlinear invariants concerns the handling of correlations. For example, if we differentiate the expression for J, we get many terms that do not appear to cancel. However, in a particular beam, such as in the example used in the numerical simulation shown in this paper, high-order correlations are absent. Furthermore, some of this lack of correlation is preserved throughout the motion. (Exactly what happens depends on the initial condition and the forces.) This means that some of the higher-order moments are actually functions of lower-order moments. The behavior of these correlations must be understood before we can derive more useful nonlinear moment invariants in the way we did for the linear case. Fortunately, the evolution of the correlations can be studied by the easy-to-use moment equations. This is because the correlations are just combinations of moments; they consist of differences of moments and the expression describing the moments in the absence of correlations.

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